31 - Lagrange Multipliers

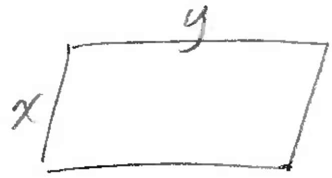
I. Picture

II. Technique

III. Examples

I. Picture.

Suppose we want to find the maximum area that can be enclosed by a rectangular fence that has a perimeter of 100ft.



objective function

$$\max \quad A = xy$$

$$\text{subject to} \quad 2x + 2y = 100$$

constraint equation

You could have used Calc I techniques to solve this, but now that we have studied multivariable functions, we can apply some new techniques to solve.

The new technique can be used to solve some problems you could not solve with Calc I techniques geogebra -

jrobbinsuk

1. 16020_Area_Lagrange-Multipliers 1
2. 16020-Area-Lagrange-Multipliers 2
3. 16020gradient

One way to encode that two functions have the same tangent line/normal line at a point (a,b) is to say that

$$f_x(a,b) = \lambda g_x(a,b)$$

and $f_y(a,b) = \lambda g_y(a,b)$

for some scalar λ

IV. Technique

To solve

$$\max \text{ or } \min \ z = f(x,y)$$ objective fctn

Subject to $g(x,y) = 0$ constraint function set $= 0$

① Solve the system of equations

$$\begin{cases} f_x(x,y) = \lambda g_x(x,y) \\ f_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases}$$

for (x,y) and λ

② Check to make sure that your max/min occurs @ (x,y) by substituting any other point on $g(x,y)=0$ into $z=f(x,y)$ to make sure you have a max or min

III. Examples

$\boxed{EX}$ max $A(x,y) = xy$.

s.t. $2x + 2y = 100 \longrightarrow 2x + 2y - 100 = 0$

$A(x,y) = xy$.

$g(x,y) = 2x + 2y - 100$

$$\begin{cases} A_x(x,y) = \lambda g_x(x,y) \\ A_y(x,y) = \lambda g_y(x,y) \\ g(x,y) = 0 \end{cases} \longrightarrow \begin{cases} y = \lambda \cdot 2 & ① \\ x = \lambda \cdot 2 & ② \\ 2x + 2y - 100 = 0 & ③ \end{cases}$$

① and ② $\rightarrow$ ③

$$2(\lambda \cdot 2) + 2(\lambda \cdot 2) - 100 = 0$$
$$8\lambda - 100 = 0$$
$$\lambda = \frac{100}{8} = 12.5$$

$\Rightarrow x = (12.5) \cdot 2 = 25$

$y = (12.5) \cdot 2 = 25$

$A(25, 25) = 625$

Another point on $g(x,y) = 0$

$(50, 0)$

$A(50,0) = 50 \cdot 0 = 0$

certainly < 625

max area $= 625$ ft^2.

$\boxed{\text{Ex}}$ (Openstax, Calc III, Ex 4.42)

Use Lagrange multipliers to find min value of $f(x,y) = x^2 + 4y^2 - 2x + 8y$ subject to the constraint $x + 2y = 7 \rightarrow x + 2y - 7 = 0$

$g(x,y) = x + 2y - 7$

$$\begin{cases} 2x - 2 = \lambda \cdot 1 & (1) \\ 8y + 8 = \lambda \cdot 2 & (2) \\ x + 2y - 7 = 0 & (3) \end{cases}$$

(1) $\lambda = 2x - 2 \rightarrow$ (2) $8y + 8 = (2x-2)\cdot 2$

$8y + 8 = 4x - 4$

$4x = 8y + 12$

$x = 2y + 3$

$\downarrow$

(3)

$2y + 3 + 2y - 7 = 0$

$4y - 4 = 0$

$y = 1 \Rightarrow x = 2(1) + 3 = 5$

$(5,1)$

$f(5,1) = 5^2 + 4(1)^2 - 2(5) + 8(1) = 27$

Any other point on $x + 2y - 7 = 0$

$(7,0)$

$f(7,0) = 7^2 + 4(0)^2 - 2(7) + 8(0) = 49 - 14 = 35$

bigger than 27.

Minimum of 27 @ (5,1).

$\boxed{Ex}$ (OpenStax, Calculus III, §4.8 # 358)

Find max and min values of $f(x,y) = x^2 y$

subject to the constraint $x^2 + 2y^2 = 6$

$$\underbrace{x^2 + 2y^2 - 6}_{g(x,y)} = 0$$

$$\begin{cases} 2xy = \lambda 2x & (1) \\ x^2 = \lambda 4y & (2) \\ x^2 + 2y^2 - 6 = 0 & (3) \end{cases}$$

Let's look @ (1)

$$2xy = \lambda 2x$$

$$2xy - 2x\lambda = 0$$

$$2x(y - \lambda) = 0$$

$$x = 0 \quad \text{or} \quad y = \lambda$$

You can<u>not</u> divide by x. This can cause you to lose solutions

Case 1: $x = 0$

$\rightarrow$ 3

$0^2 + 2y^2 - 6 = 0$

$y^2 = 3$

$y = \pm\sqrt{3}$

Points to consider

$(0, \sqrt{3})$

$(0, -\sqrt{3})$

Case 2: $y = \lambda$

2 $\Rightarrow x^2 = \lambda 4y$

$x^2 = y4y$

$x^2 = 4y^2$

$\rightarrow$ 3

$4y^2 + 2y^2 - 6 = 0$

$6y^2 - 6 = 0$

$y^2 = 1$

$y = \pm 1$

$y = \lambda = 1$

$x^2 + 2(1)^2 - 6 = 0$

$x^2 - 4 = 0$

$x = \pm 2$

$y = \lambda = -1$

$x^2 + 2(-1)^2 - 6 = 0$

$x^2 - 4 = 0$

$x = \pm 2$

Points to consider

$(2, 1)$ $(2, -1)$

$(-2, 1)$ $(-2, -1)$

(x, y)	$f(x, y) = x^2 y$
$(0, \sqrt{3})$	0
$(0, -\sqrt{3})$	0
$(2, 1)$	4
$(2, -1)$	-4
$(-2, 1)$	4
$(-2, -1)$	-4

max val $= 4$ @ $(\pm 2, 1)$

min val $= -4$ @ $(\pm 2, -1)$

$\boxed{EX}$ Find the max value of

$f(x,y) = \ln(3x^2y)$

subject to $4x^2 + 5y^2 = 16$

$f(x) = \ln(3) + 2\ln(x) + \ln(y)$

$g(x) = 4x^2 + 5y^2 - 16$

$$\begin{cases} \dfrac{2}{x} = \lambda 8x & (1) \\ \dfrac{1}{y} = \lambda 10y & (2) \\ 4x^2 + 5y^2 - 16 = 0 & (3) \end{cases}$$

<u>Note:</u> $\lambda \neq 0$
$x \neq 0$
$y \neq 0$

(1) $2 = \lambda 8x^2$

$x^2 = \dfrac{2}{8\lambda} = \dfrac{1}{4\lambda}$

(2) $1 = \lambda 10y^2$

$y^2 = \dfrac{1}{10\lambda}$

$\rightarrow$ (3) $4 \cdot \dfrac{1}{4\lambda} + 5 \cdot \dfrac{1}{10\lambda} - 16 = 0$

$\dfrac{1}{\lambda} + \dfrac{1}{2\lambda} = 16$

$\dfrac{3}{2\lambda} = 16$

$$\frac{2\lambda}{3} = \frac{1}{16}$$

$$\lambda = \frac{3}{32}.$$

$$x^2 = \frac{1}{4\left(\frac{3}{32}\right)} = \frac{32}{12} = \frac{8}{3} \qquad x = \pm\sqrt{\frac{8}{3}}$$

$$y^2 = \frac{1}{10\left(\frac{3}{32}\right)} = \frac{32}{30} = \frac{16}{15} \qquad y = \pm\sqrt{\frac{16}{15}} = \pm\frac{4}{\sqrt{15}}$$

Since $f(x,y) = \ln\big(\underbrace{\overset{>0}{\widetilde{3x^2}}\,y}_{\text{must be } >0}\big) \Rightarrow y$ cannot be negative

Points to consider:

(x,y)	$f(x,y) = \ln(3x^2y)$
$\left(\sqrt{\frac{8}{3}}, \sqrt{\frac{16}{15}}\right)$	$\ln\left(3\left(\frac{8}{3}\right)\sqrt{\frac{16}{15}}\right) \approx 2.1117$
$\left(-\sqrt{\frac{8}{3}}, \sqrt{\frac{16}{15}}\right)$	$\ln\left(3\left(\frac{8}{3}\right)\sqrt{\frac{16}{15}}\right) \approx$
$\cancel{\left(\sqrt{\frac{8}{3}}, -\sqrt{\frac{16}{15}}\right)}$	
$\cancel{\left(-\sqrt{\frac{8}{3}}, -\sqrt{\frac{16}{15}}\right)}$	

Another point on $4x^2+5y^2-16=0$ and in domain of f.

$x=1$

$4+5y^2-16=0$

$5y^2=12$

$y=\sqrt{\frac{12}{5}}$

$(1, \sqrt{\frac{12}{5}})$

$f(1, \sqrt{\frac{12}{5}}) = \ln(3 \cdot 1^2 \sqrt{\frac{12}{5}}) \approx 1.5363$

max of about 2.1117 @ $(\pm\sqrt{\frac{8}{3}}, \sqrt{\frac{16}{15}})$